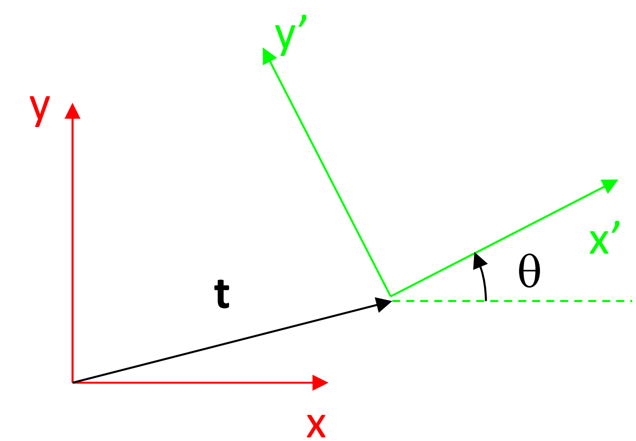
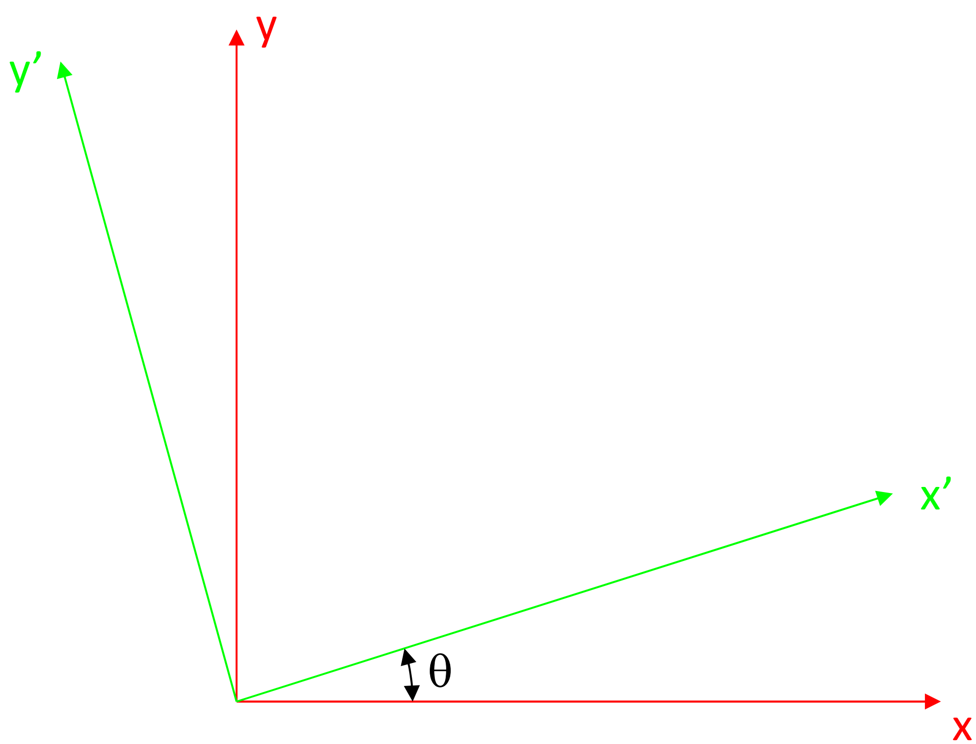
2D-2D Coordinate Transforms

2D Rigid Frame Transformations

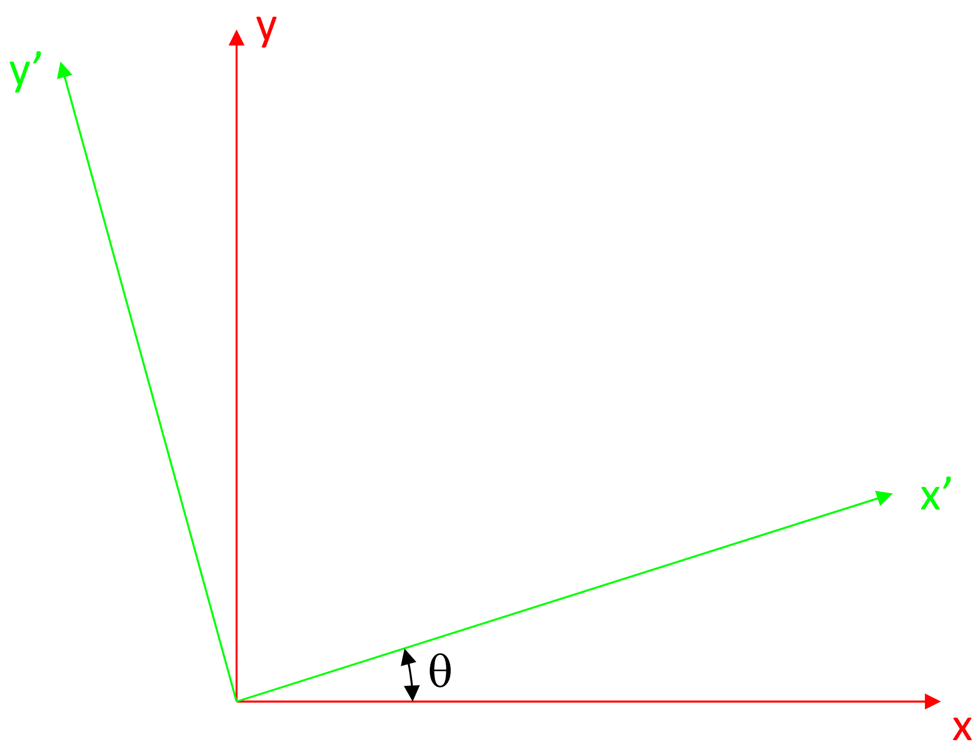
- The pose of one 2D frame with respect to another is described by
  - Translation vector **t**=$(\Delta x, \Delta y)^T$
  - Rotation angle θ
    - Rotation can also be represented as a 2x2 matrix **R**
- Object shape and size is preserved
- Number of degrees of freedom for a 2D rigid transformation?



Rotations in 2D



Rotations in 2D



$$\begin{pmatrix} x \\ y \end{pmatrix} = \mathbf{R} \begin{pmatrix} x' \\ y' \end{pmatrix}$$

$$\mathbf{R} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

- Note: **R** is orthonormal
  - Rows, columns are orthogonal ($\mathbf{r}_1 \cdot \mathbf{r}_2 = 0$, $\mathbf{c}_1 \cdot \mathbf{c}_2 = 0$)
  - Transpose is the inverse; $\mathbf{R}\mathbf{R}^T = \mathbf{I}$
  - Determinant is $|\mathbf{R}| = 1$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \mathbf{R}^{\mathrm{T}} \begin{pmatrix} x \\ y \end{pmatrix}$$

Homogeneous Coordinates

- Points can be represented using homogeneous coordinates
  - This simply means to append a 1 as an extra element
  - If the 3rd element becomes ≠ 1, we divide through by it

$$\tilde{\mathbf{x}} = \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} sx \\ sy \\ s \end{pmatrix}$$

- Effectively, vectors that differ only by scale are considered to be equivalent
- This simplifies transform equations; instead of

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \mathbf{R} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} t_x \\ t_y \end{pmatrix} \qquad \mathbf{x}' = \mathbf{R}\mathbf{x} + \mathbf{t}$$

- we have

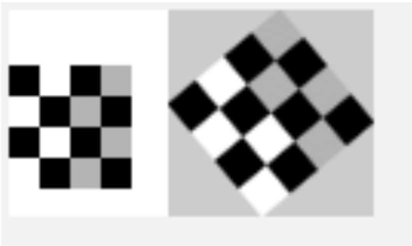
$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{pmatrix} \mathbf{R} & \mathbf{t} \\ 0 \;\; 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} \qquad \widetilde{\mathbf{x}'} = \mathbf{H}\tilde{\mathbf{x}}$$

$\tilde{\mathbf{x}} \in P^2$, where $P^2 = R^3 - (0,0,0)$ is called a 2D projective space

Other 2D-2D Transforms

- Scaled (similarity) transform
  - preserves angles but not distances

$$\begin{pmatrix} x_B \\ y_B \\ 1 \end{pmatrix} = \begin{pmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta & t_x \\ \sin\theta & \cos\theta & t_y \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_A \\ y_A \\ 1 \end{pmatrix}$$



- Affine transform
  - Models rotation, translation, scaling, shearing, and reflection

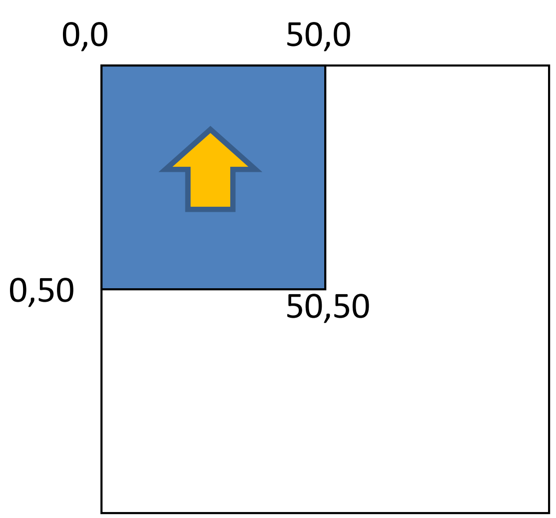
$$\begin{pmatrix} x_B \\ y_B \\ 1 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & t_x \\ a_{21} & a_{22} & t_y \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_A \\ y_A \\ 1 \end{pmatrix}$$

pairs of corresponding points needed?

Example

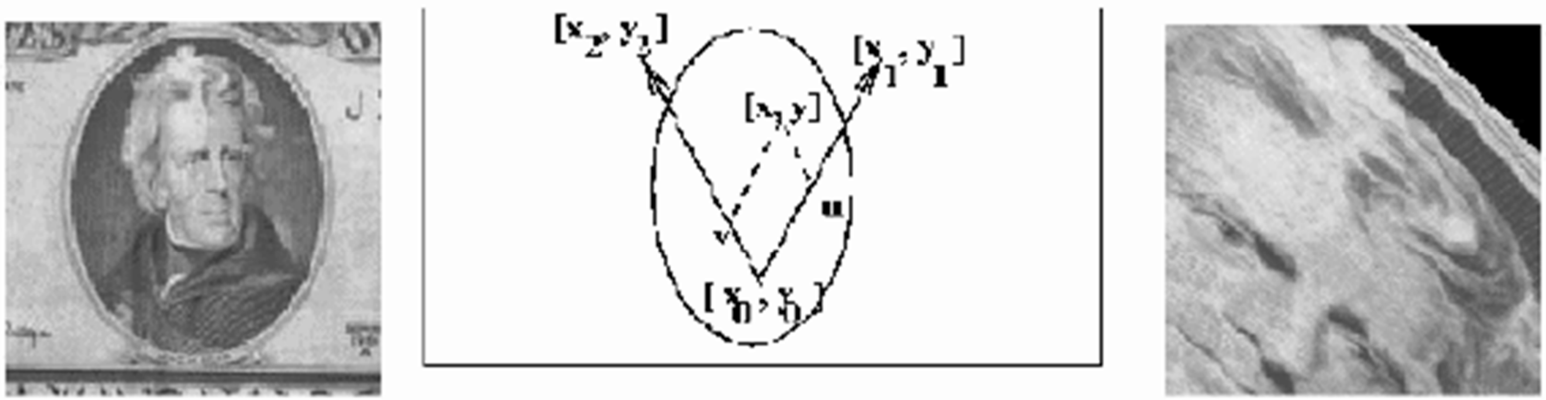
- Image "A" is modified by the affine transform below. Sketch image "B"

$$\begin{pmatrix} x_B \\ y_B \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0.25 & 1.5 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_A \\ y_A \\ 1 \end{pmatrix}$$



A

Example Affine Warp



Distorted face of Andrew Jackson extracted from a $20 bill by defining an affine mapping with shear.

from http://www.cse.msu.edu/~stockman/CV

Projective Transform (Homography)

- Most general type of linear 2D-2D transform
- A 3x3 matrix, with an unknown scale factor

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} x_A \\ y_A \\ 1 \end{pmatrix}, \quad x_B = x_1 / x_3,\ y_B = x_2 / x_3$$

- It transforms one projective space to another

- Equivalently, it maps points from one plane to another plane

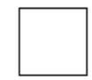
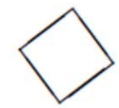
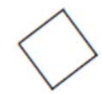
Transformation	Matrix	# DoF	Preserves	Icon
translation	$\begin{bmatrix} \boldsymbol{I} \mid \boldsymbol{t} \end{bmatrix}_{2\times 3}$	2	orientation	
rigid (Euclidean)	$\begin{bmatrix} \boldsymbol{R} \mid \boldsymbol{t} \end{bmatrix}_{2\times 3}$	3	lengths	
similarity	$\begin{bmatrix} s\boldsymbol{R} \mid \boldsymbol{t} \end{bmatrix}_{2\times 3}$	4	angles	
affine	$\begin{bmatrix} \boldsymbol{A} \end{bmatrix}_{2\times 3}$	6	parallelism	
projective	$\begin{bmatrix} \tilde{\boldsymbol{H}} \end{bmatrix}_{3\times 3}$	8	straight lines	

Table 2.1 Hierarchy of 2D coordinate transformations. Each transformation also preserves the properties listed in the rows below it, i.e., similarity preserves not only angles but also parallelism and straight lines. The 2×3 matrices are extended with a third $[\mathbf{0}^T\ 1]$ row to form a full 3×3 matrix for homogeneous coordinate transformations.

From Szeliski, Computer Vision: Algorithms and Applications